Systems of logic for necessity

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Abstract: This paper discusses a theory of modal predicate logic, which is based on S5. This paper gives four ways to define this theory. The first way is to define a system which tells whether a context supports a formula or not. In this way we evaluate a formula with not a model but a context. This theory distinguish occasional equality from necessary equality, as is discussed with the problem of referentially opaque context. The aim of this paper is to observe the mathematical properties of this theory. The second way is a derivation system of Hirbert-style, which is given by adding new axiom schemata to the rules of S5. The third is a derivation system of Gentzen-style, where a sequent is not a sequence of formulae but a table of formulae. his system satisfies cut elimination. The fourth is a kind of possible-world semantics. In the fourth way, a formula is valid when the formula is true in each partial abstraction models. This paper shows the equivalence of these four ways, that is, these four ways define the same theory.

Keywords: predicate modal logic, S5, referentially opaque context, Hirbert-style system, Gentzen-style system, cut elimination, possible-world semantics

1 Introduction

There is a curious system of modal logic which appears in a literature [T]. The system is a kind of semantical system, which tells whether a context supports a formula or not. we call this system a system of *logic for necessity.*

A context is a pair of formulae $\langle C, D\rangle$ which have no modal symbols and are consistent to each other, that is, $\nvdash \neg(C \wedge D)$ in classical logic. The left formula C of the context $\langle C, D\rangle$ is called a permeant context and the right formula D is called a sheltered context. We write $C, D \models_{\mathrm{L}} P$ when the contest $\langle C, D\rangle$ supports the formula P. The relation $\models_{\mathrm{L}}$ is defined by induction on the construction of the formula. The precise formal definition appears in Def. 3.4. The intuitive meaning of $C, D \models_{\mathrm{L}} P$ is the following. The modal operator $\Box$ shelters the sheltered context, but cannot shelter the permeant context. For example, if P is a classical formula, then $C, D \models_{\mathrm{L}} P$ iff $\vdash C \supset D \supset P$ in classical logic, and $C, D \models_{\mathrm{L}} \Box P$ iff $\vdash C \supset P$ in classical logic. We write $\models_{\mathrm{L}} P$ and call P a valid formula when $C, D \models_{\mathrm{L}} P$ for each context $\langle C, D\rangle$. Then the set of valid formula is regarded as a theory of modal logic. For example, if $\models_{\mathrm{L}} P \supset Q$ and $\models_{\mathrm{L}} P$, then $\models_{\mathrm{L}} Q$ (Cor. 3.13). The propositional part of $\models_{\mathrm{L}}$ is equivalent to S5 (Cor. 4.6), although $\not\models_{\mathrm{L}} x = y \supset \Box x = y$.

There is some background discussion of this system from the view point of logic in the literature [T]. But, the aim of this paper is not such a discussion from the logical view point. This paper aims at discussing only the mathematical properties of this system, with comparing other three systems.

This paper introduce three other systems of modal logic. The second system is a derivation system of Hirbert-style, which is given by adding a new axiom schema to the rules of S5. The third is a derivation system of Gentzen-style, where a sequent is not a sequence of formulae but a table of formulae. This system satisfies cut elimination. The fourth is a kind of possible-world semantics. In the third way, a formula is valid when the formula is true in each partial abstraction models. This paper shows the equivalence of these four ways, that is, these four ways define the same theory.

2 Language of modal predicate logic

Definition 2.1 (Language) A set Σ consists of finite signatures. Some of them are ***predicates*** and the others are ***function symbols.*** Each $\sigma \in \Sigma$ has its arity in $\{0, 1, 2, ...\}$. A function symbol of arity 0 is regarded as a constant. A set V is the set of infinitely many ***variables.*** There are four ***logical symbols,*** which are conjunction $\wedge$, negation $\neg$, universal quantifier $\forall$ and modality of necessity $\Box$.

Terms are generated in the ordinary way by the function symbols in Σ and variables in V. *Formulae* are generated in the ordinary way by the terms, predicates in Σ, and four logical symbols.

The set Σ has at least one predicates. The set Σ either may have equality '=' as a binary predicate, or not. The sets Σ and V is ordered by a linear ordering. This ordering is used to define the lexicographical ordering over the terms and the formulae.

The notions of ***free variables, bound variables, renaming*** of variables, and ***substitution*** of variables with terms are defined in the ordinary way.

Notation 2.2 If '=' $\in \Sigma$, then we write the equations in the usual way as $t = t'$ instead of $=(t, t')$.

Definition 2.3 (Classical formulae) A formula without $\Box$ is called a ***classical formula.***

Notation 2.4 The language does not have implication, disjunction, existential quantifier, nor possibility as primitives. We will use such logical symbols as abbreviations.

1. $P \supset Q :\equiv \neg(P \wedge \neg Q)$
2. $P \vee Q :\equiv \neg(P \wedge Q)$

3. $P \supset\subset Q :\equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$

4. $\exists x.P :\equiv \neg\forall x.\neg P$

5. $\Diamond P :\equiv \neg\Box\neg P$

Notation 2.5 In this paper, the connective powers of logical symbols are listed in the order as $\Box, \Diamond, \neg, \wedge, \vee, \supset, \supset\subset, \forall, \exists$. The connective powers of $\Box$, $\Diamond$ and $\neg$ are strongest and those of $\forall$ and $\exists$ are weakest. The symbols $\wedge$ and $\vee$ are left associative and the symbol $\supset$ is right associative.

Definition 2.6 (Classical logic) For a classical formula P, we write $\vdash_{\mathrm{C}} P$ if P is derivable in the classical logic. Note that if '=' $\in \Sigma$ then the rules of equalities, namely the transitivity, are derivable in this classical logic.

3 Logic for necessity

Definition 3.1 (Formula of truth) We write $\top$ for the formula $\neg(\neg P \wedge P)$, where P is the first closed classical formula in the lexicographical ordering. This plays the role of the representative of true formulae.

Definition 3.2 (Consistency) Let P and Q be classical formulae, which may be open or closed. Then the formula P is *consistent* iff $\not\vdash_{\mathrm{C}} \neg P$. The formula P is *consistent to* Q iff $\not\vdash_{\mathrm{C}} \neg(P \wedge Q)$.

Definition 3.3 (Context, permeant context and sheltered context) Let C and D be classical formulae which are consistent to each other. Then the pair $\langle C, D\rangle$ is called a *context*. In the context $\langle C, D\rangle$, the left formula C is called the *permeant context*, and the right formula D is called the *sheltered context*.

If the sheltered context is $\top$, then we sometimes write $\langle C, -\rangle$ instead of $\langle C, \top\rangle$.

Definition 3.4 (System of logic for necessity) For a formula P and a context $\langle C, D\rangle$, the relation $C, D \models_{\mathrm{L}} P$ is defined in the induction on P as below.

1. If P has no logical symbols, then $C, D \models_{\mathrm{L}} P$ iff $\vdash_{\mathrm{C}} (C \wedge D) \supset P$.

2. $C, D \models_{\mathrm{L}} P \wedge Q$ iff $C, D \models_{\mathrm{L}} P$ and $C, D \models_{\mathrm{L}} Q$.

3. $C, D \models_{\mathrm{L}} \neg P$ iff for an arbitrary classical formula D' which is consistent to $C \wedge D$, it holds that $C, D \wedge D' \not\models_{\mathrm{L}} P$.

4. $C, D \models_{\mathrm{L}} \forall x.P$ iff $C, D \models_{\mathrm{L}} P[t/x]$ for all the terms t.

5. $C, D \models_{\mathrm{L}} \Box P$ iff $C, \top \models_{\mathrm{L}} P$

We sometimes write $C, - \models_{\mathrm{L}} P$ for $C, \top \models_{\mathrm{L}} P$.

Definition 3.5 (Valid formula) For a formula P, we write $\models_{\mathrm{L}} P$ if $C, D \models_{\mathrm{L}} P$ for each context $\langle C, D\rangle$. we call P a *valid formula* when $\models_{\mathrm{L}} P$.

Remark 3.6 it is because of the rule $C, D \models_{\mathrm{L}} \Box P$ iff $C, \top \models_{\mathrm{L}} P$, that C is called permeant and D is called sheltered.

Definition 3.7 (Weakest sheltered context) For a classical formula C and a formula P, the *weakest sheltered context* $\delta_C(P)$ is a classical formula defined in the induction on P as below.

1. If P has no logical symbols, then $\delta_C(P) :\equiv C \supset P$.
2. $\delta_C(P \wedge Q) :\equiv \delta_C(P) \wedge \delta_C(Q)$.
3. $\delta_C(\neg P) :\equiv \neg(C \wedge \delta_C(P))$.
4. $\delta_C(\forall x.P) :\equiv \forall y.\delta_C(P[y/x])$ where y is not a free variable in C nor in D.
5. $\delta_C(\Box P) :\equiv \top$ iff $C \vdash_{\mathrm{C}} \delta_C(P)$, and $\delta_C(\Box P) :\equiv \neg C$ iff $C \not\vdash_{\mathrm{C}} \delta_C(P)$.

Remark 3.8 All the free variable of $\delta_C(P)$ are some free variables in C and P.

Remark 3.9 The formula $\delta_C(P \vee Q)$ is equivalent to $\delta_C(P) \vee \delta_C(Q)$. The formula $\delta_C(\Diamond P)$ is equivalent to $\neg C$ if $C \vdash_{\mathrm{C}} \neg\delta_C(P)$, and $\delta_C(\Diamond P)$ is equivalent to $\top$ if $C \not\vdash_{\mathrm{C}} \neg\delta_C(P)$.

Proposition 3.10 *Let C be classical formula and P be a formula. Then* $\vdash_{\mathrm{C}} \neg C \supset \delta_C(P)$

Proposition 3.11 *Let C and P be classical formulae.*
1. $\vdash_{\mathrm{C}} \delta_C(P) \supset\subset C \supset P$.
2. If $\vdash_{\mathrm{C}} C \supset P$ *then* $\delta_C(\Box P) \equiv \top$, *and if* $\not\vdash_{\mathrm{C}} C \supset P$ *then* $\delta_C(\Box P) \equiv \neg C$
3. If $\not\vdash_{\mathrm{C}} \neg(C \wedge P)$ *then* $\delta_C(\Diamond P) \equiv \top$, *and if* $\vdash_{\mathrm{C}} \neg(C \wedge P)$ *then* $\delta_C(\Diamond P) \equiv \neg C$

Lemma 3.12 *Let P be a formula and C and D be classical formulae. Then* $\vdash_{\mathrm{C}} D \supset \delta_C(P)$ *iff either* $\vdash_{\mathrm{C}} \neg(C \wedge D)$ *or* $C, D \models_{\mathrm{L}} P$.

Corollary 3.13 *Let P and Q be formulae and $\langle C, D\rangle$ be a context. If $C, D \models_{\mathrm{L}} P \supset Q$ and $C, D \models_{\mathrm{L}} P$, then $C, D \models_{\mathrm{L}} Q$.*

Corollary 3.14 (Consistency) *The set of valid formulae of $\models_{\mathrm{L}}$ is consistent. Moreover, it is a conservative extension of classical logic.*

Notation 3.15 We write $F[X_1, ..., X_n]$ for a formula which is constructed from propositional variables $X_1, ..., X_n$ with logical symbols $\neg$, $\wedge$ and $\Box$. Let $P_1, ...,$ P_n be formulae. Then we write $F[P_1, ..., P_n]$ for the formula $(F[X_1, ..., X_n])$ $[P_1/X_1, ..., P_n/X_n]$ which is given by the substitution of X_i's with P_i's in $F[X_1,$ $..., X_n]$.

Lemma 3.16 *If $F[X_1, ..., X_n]$ does not contain $\Box$, then $\delta_C(F[P_1, ..., P_n])$ is equivalent to $F[\delta_C(P_1), ..., \delta_C(P_n)]$ in classical logic.*

Corollary 3.17 *If $F[X_1, ..., X_n]$ is a classical formula which is a theorem of classical logic, then* $\models_{\mathrm{L}} F[P_1, ..., P_n]$.

Remark 3.18 According to Cor. 3.13, Cor. 3.14, and Cor. 3.17, the set of valid formulae is regarded as a theory of modal logic.

Remark 3.19 The followings hold.
$\models_{\mathrm{L}} (\forall x.\Box P) \supset \Box\forall x.P$ (Barcan formula)
$\models_{\mathrm{L}} (\forall x.\Diamond P) \supset \Diamond\forall x.P$
$\models_{\mathrm{L}} (\exists x.P \wedge \Diamond\exists x.Q) \supset \exists x.P \wedge \Diamond Q$, if P is a classical formula.
$\not\models_{\mathrm{L}} x = y \supset \Box x = y$

Remark 3.20 There has been a lot of discussion over the failure of $x = y \supset \Box x = y$, which appears in the problem of referentially opaque context ([Q], and also Sec. 2, Chap. 4 in [N]). We do not discuss such problems in this paper.

4 Propositional part of S5

Definition 4.1 (S5) The derivation system S5 is defined as the following rules:

Axiom: $\dfrac{}{P}$, where P is one of the following:

- **Classical theorem:** $F[P_1, ..., P_n]$, where $F[X_1, ..., X_n]$ is a theorem of classical logic.
- **Instantiation:** $(\forall x.P) \supset P[t/x]$
- **Universality sift:** $(\forall x.P \supset Q) \supset P \supset \forall x.Q$, where x is not free in P.
- **Equality:** $x = y \supset P \supset P[y/x]$, where P is a classical formula, if the language has the symbol '='.
- **K:** $\Box(P \supset Q) \supset \Box P \supset \Box Q$
- **T:** $\Box P \supset P$
- **D:** $\Diamond\top$
- **5:** $\Diamond\Box P \supset \Box P$

Implication elimination: $\dfrac{P \supset Q \qquad P}{Q}$

Universality introduction: $\dfrac{P}{\forall x.P}$

Necessity introduction: $\dfrac{P}{\Box P}$

We write $\vdash_{\mathrm{S5}} P$ if P is a theorem of S5.

Remark 4.2 The formula $x = y \supset \Box x = y$ is not a theorem in the system S5 defined here, although the system S5 is sometimes defined such as $x = y \supset \Box x = y$ is its theorem.

Definition 4.3 (Propositional Formula) A *propositional formula* is a formula without the symbols $\forall$ and $=$.

Definition 4.4 (S5P) The derivation system S5P, which is the propositional part of S5, is defined as S5 with the restriction such that all of the formulae in the proof are propositional formula. Thus, none of the rules of equality, instantiation, universality sift, nor universality introduction appears in the proofs. We write $\vdash_{\mathrm{S5P}} P$ if P is a theorem of S5P.

Lemma 4.5 *For each formula P, If $\vdash_{\mathrm{S5}} P$, then $\models_{\mathrm{L}} P$.*

Corollary 4.6 (Global Soundness of S5P) *For each propositional formula P, If $\vdash_{\mathrm{S5P}} P$, then $\models_{\mathrm{L}} P$.*

Corollary 4.7 (Casewise soundness of S5P) *Let P be a propositional formula. Let C and D be classical propositional formulae which are consistent to each other. If there are some classical propositional formulae $E_1, E_2, ..., E_n$ such that each E_i is consistent to C and $\vdash_{\mathrm{S5P}} \Box C \supset D \supset \Diamond E_1 \supset \cdots \Diamond E_n \supset P$, then $C, D \models_{\mathrm{L}} P$.*

Lemma 4.8 (Normalisation of S5P) *Let P be a propositional formula. Then there are classical propositional formulae*

$$C_1, D_1, E_{11}, ..., E_{1m_1},$$
$$C_2, D_2, E_{21}, ..., E_{2m_2},$$
$$\ldots$$
$$C_n, D_n, E_{n1}, ..., E_{nm_n}$$

such that S5P derives

$$P \supset\subset \bigwedge_{i=1,..,n} \Diamond C_i \vee D_i \vee \bigvee_{j=1,..,m_i} \Box E_{ij}$$

As the duality, there also exists a disjunctive-conjunctive normal form.

Lemma 4.9 (Casewise completeness of S5P) *Let P be a propositional formula. Let C and D be classical propositional formulae which are consistent to each other. If $C, D \models_{\mathrm{L}} P$, then there are some classical propositional formulae $E_1, E_2, ..., E_n$ such that each E_i is consistent to C and $\vdash_{\mathrm{S5P}} \Box C \supset D \supset \Diamond E_1 \supset \cdots \Diamond E_n \supset P$,*

Lemma 4.10 (Global completeness of S5P) *Let P be a propositional formula. If $\models_{\mathrm{L}} P$, then $\vdash_{\mathrm{S5P}} P$.*

5 Distributive modality

Definition 5.1 (S5D) The derivation system of *distributive modality of S5* is defined by adding the following axiom schema to the system S5.

- **Distributivity:**
  $(\Box\exists x.C) \wedge (\exists x.D) \wedge (\Diamond\exists x.E_1) \wedge ... \wedge (\Diamond\exists x.E_n)$
  $\supset \exists x.\ \Box(C \vee D \vee E_1 \vee ... \vee E_n) \wedge D \wedge \Diamond E_1 \wedge ... \wedge \Diamond E_n$
  where all of C, D, E_1, ..., E_n are classical formulae.

We call this system S5D. We write $\vdash_{\mathrm{S5D}} P$ when P is derivable in this system.

Remark 5.2 The axiom of distributivity is equivalent to a little more complicated form:

$$((\Box\exists x.C \vee D \vee \bigvee_i E_i) \wedge \exists x.D \wedge \bigwedge_i \Diamond\exists x.E_i) \supset \exists x.\Box(C \vee D \vee \bigvee_i E_i) \wedge D \wedge \bigwedge_i \exists x.\Diamond E_i$$

The inverse direction of this form

$$(\exists x.\Box(C \vee D \vee \bigvee_i E_i) \wedge D \wedge \bigwedge_i \exists x.\Diamond E_i) \supset (\Box\exists x.C \vee D \vee \bigvee_i E_i) \wedge \exists x.D \wedge \bigwedge_i \Diamond\exists x.E_i$$

is already a theorem of S5.

Lemma 5.3 (Normalisation of S5D) *Let P be an arbitrary formula. Then there are classical formulae*

$$C_1, D_1, E_{11}, ..., E_{1m_1},$$
$$C_2, D_2, E_{21}, ..., E_{2m_2},$$
$$...$$
$$C_n, D_n, E_{n1}, ..., E_{nm_n}$$

such that S5D derives

$$P \supset\subset \bigwedge_{i=1,..,n} \Diamond C_i \vee D_i \vee \bigvee_{j=1,...,m_i} \Box E_{ij}$$

As the duality, there also exists a disjunctive-conjunctive normal form.

Remark 5.4 Let P be a theorem of S5. We make a formula P' by erasing all the occurrences of $\Box$ in P. Then P' is a theorem of classical logic. It does not hold for S5D. For example, let A and B be classical formulae. Then $(\exists x.\Diamond A) \wedge (\exists x.\Diamond B) \supset \exists x.\Diamond A \wedge \Diamond B$ is a theorem of S5D, although, of course, $(\exists x.A) \wedge (\exists x.B) \supset \exists x.A \wedge B$ is not a theorem of classical logic.

Lemma 5.5 (Global soundness of S5P) *For each formula P, if $\vdash_{\mathrm{S5D}} P$ then $\models_{\mathrm{L}} P$.*

Corollary 5.6 *The system S5D is consistent. Especially, $\not\vdash_{\mathrm{S5P}} \forall xy.\, x = y$.*

Corollary 5.7 *S5D is a conservative extension of S5P.*

Corollary 5.8 (Casewise soundness of S5D) *Let P be a formula and $\langle C, D\rangle$ be a context. If there are some classical formulae $E_1, E_2, ..., E_n$ such that each E_i is consistent to C and $\vdash_{\mathrm{S5D}} \Box C \supset D \supset \Diamond E_1 \supset \cdots \Diamond E_n \supset P$, then $C, D \models_{\mathrm{L}} P$.*

Theorem 5.9 (Casewise completeness of S5D) *Let P be a formula and $\langle C, D\rangle$ be a context. If $C, D \models_{\mathrm{L}} P$, then there are some classical formulae $E_1, E_2, ..., E_n$ such that each E_i is consistent to C and $\vdash_{\mathrm{S5D}} \Box C \supset D \supset \Diamond E_1 \supset \cdots \Diamond E_n \supset P$.*

Theorem 5.10 (Global completeness of S5D) *For each formula P, if $\models_{\mathrm{L}} P$ then $\vdash_{\mathrm{S5D}} P$.*

Remark 5.11 The system S5P satisfies the substitution on formulae, which is the following property. Let P, Q be formulae and p be a predicate of arity 0. The formula $P[Q/p]$ is made by substitution of p with Q in P. If $\vdash_{\mathrm{S5P}} P$ then $\vdash_{\mathrm{S5P}} P[Q/p]$.

The theory of classical predicate logic also satisfies the substitution on predicates, which is the following property. Let $P, R[x_1, ..., x_n$ be a formulae and p be a predicate of arity n. The formula $P[R/p]$ is made by substitution of each occurrence of $p(t_1, ..., t_n$ with $R[t_1, ..., t_n]$ in P. If $\vdash_{\mathrm{C}} P$ then $\vdash_{\mathrm{C}} P[R/p]$.

However, the systems S5 and S5D do not satisfies the substitution on predicates. That is because the axioms of equation and distributivity are sensitive of modality.

6 Gentzen-style derivation system

Remark 6.1 Hereafter, we require the language Σ to have equation '='.

Definition 6.2 (Sequent) A *sequent* is a table formed of sequences of formulae as below.

$$\begin{array}{|c|c|}
\hline
A_{01}, ..., A_{0l_0} & B_{01}, ..., B_{0m_0} \\
\hline
A_{11}, ..., A_{1l_1} & B_{11}, ..., B_{1m_1} \\
\cdots & \cdots \\
A_{n1}, ..., A_{nl_n} & B_{n1}, ..., B_{nm_n} \\
\hline
E_1, ..., E_{l_{n+1}} & \\
\cline{1-1}
\end{array}$$

where each of A_{ij} and B_{ij} is a formula, and each E_i is an equation such as $t = t'$.

Some of sequences "$A_{i1}, ..., A_{il_i}$", "$B_{i1}, ..., B_{im_i}$", and "$E_1, ..., E_{l_{n+1}}$" may be empty. The number n, which is the number of the rows of the middle part, may be 0. In such case, the middle part would be empty, such as

$$\begin{array}{|c|c|}
\hline
A_1, ..., A_l & B_1, ..., B_m \\
\hline
\hline
E_1, ..., E_{l'} & \\
\cline{1-1}
\end{array}$$

Notation 6.3 Let Γ be a sequence of formulae such as "$A_1, ..., A_n$". We write $\neg\Gamma$ for the sequence "$\neg A_1, ..., \neg A_n$". We write $\wedge\Gamma$ for the formulae "$A_1 \wedge ... \wedge$

A_n", and $\vee\Gamma$ for the formulae "$A_1 \vee ... \vee A_n$", If Γ is empty, then $\wedge\Gamma$ stands for $\top$ and $\vee\Gamma$ stands for $\neg\top$.

Definition 6.4 (Interpretation) Let $\Gamma_0, ..., \Gamma_n, \Delta_0, ..., \Delta_n$ be sequences of formulae, and E be a sequence of equations. Let S be a sequent such as

$$S = \begin{array}{|c|c|} \hline \Gamma_0 & \Delta_0 \\ \hline \Gamma_1 & \Delta_1 \\ \cdots & \cdots \\ \Gamma_n & \Delta_n \\ \hline \multicolumn{1}{|c|}{E} \\ \cline{1-1} \end{array}$$

Then, the interpretation of S is the formula $[\![S]\!]$ such that

$$\begin{array}{ll} [\![S]\!] \equiv \neg(& \\ & \Box((\wedge\Gamma_0) \wedge (\wedge\neg\Delta_0)) \\ & \wedge \left(\bigwedge_{i=1,...,n} \Diamond((\wedge\Gamma_i) \wedge (\wedge\neg\Delta_i)) \right) \\ & \wedge\Box \left((\wedge E) \vee \bigvee_{i=1,...,n} (\wedge\Gamma_i) \wedge (\wedge\neg\Delta_i) \right) \\ ) & \end{array}$$

Remark 6.5 The intuitive meaning of $[\![S]\!]$ is the following. If $[\![S]\!]$ does not hold, then:

1. $(\wedge\Gamma_0) \wedge (\wedge\neg\Delta_0)$ must hold.
2. $(\wedge\Gamma_i) \wedge (\wedge\neg\Delta_i)$ may hold for each $i = 1, ..., n$.
3. $\wedge E$ must hold if none of $(\wedge\Gamma_i) \wedge (\wedge\neg\Delta_i)$ for $i = 1, ..., n$ holds.

Definition 6.6 (Deduction rules) Unfortunately we cannot put the whole rules in the main sections because of the limit of pages. The whole rules appear in the appendix. The important rules are the rules on modality and variable elimination. The left rules of modality are:

$$\frac{\begin{array}{|r|c|} \hline \Gamma_0 & \Delta_0 \\ \hline \cdots & \cdots \\ P, \Gamma_i & \Delta_i \\ \cdots & \cdots \\ \hline \multicolumn{1}{|c|}{E} \\ \cline{1-1} \end{array}}{\begin{array}{|r|c|} \hline P, \Gamma_0 & \Delta_0 \\ \hline \cdots & \cdots \\ \Gamma_i & \Delta_i \\ \cdots & \cdots \\ \hline \multicolumn{1}{|c|}{E} \\ \cline{1-1} \end{array}}, \quad \frac{\begin{array}{|r|c|} \hline \Gamma_0 & \Delta_0 \\ \hline \cdots & \cdots \\ \Gamma_i & \Delta_i \\ \cdots & \cdots \\ \hline \multicolumn{1}{|c|}{t = t', E} \\ \cline{1-1} \end{array}}{\begin{array}{|r|c|} \hline t = t', \Gamma_0 & \Delta_0 \\ \hline \cdots & \cdots \\ \Gamma_i & \Delta_i \\ \cdots & \cdots \\ \hline \multicolumn{1}{|c|}{E} \\ \cline{1-1} \end{array}}, \quad \frac{\begin{array}{|r|c|} \hline P, \Gamma_0 & \Delta_0 \\ \hline \cdots & \cdots \\ \Gamma_i & \Delta_i \\ \cdots & \cdots \\ \hline \multicolumn{1}{|c|}{E} \\ \cline{1-1} \end{array}}{\begin{array}{|r|c|} \hline \Gamma_0 & \Delta_0 \\ \hline \cdots & \cdots \\ \Box P, \Gamma_i & \Delta_i \\ \cdots & \cdots \\ \hline \multicolumn{1}{|c|}{E} \\ \cline{1-1} \end{array}},$$

and the right rules of modality are:

$$\frac{\begin{array}{|c|c|}\hline \Gamma_0 & \Delta_0 \\ \hline \Gamma_1 & \Delta_1 \\ \cdots & \cdots \\ \Gamma_n & \Delta_n \\ E & P \\ \hline E & \\ \cline{1-1} \end{array}}{\begin{array}{|c|c|}\hline \Gamma_0 & \Delta_0, \Box P \\ \hline \Gamma_1 & \Delta_1 \\ \cdots & \cdots \\ \Gamma_n & \Delta_n \\ \hline E & \\ \cline{1-1} \end{array}}, \quad \frac{\begin{array}{|c|c|}\hline \Gamma_0 & \Delta_0, P \\ \hline \cdots & \cdots \\ \Gamma_i & \Delta_i \\ \cdots & \cdots \\ \hline E & \\ \cline{1-1} \end{array}}{\begin{array}{|c|c|}\hline \Gamma_0 & \Delta_0 \\ \hline \cdots & \cdots \\ \Gamma_i & \Delta_i, \Box P \\ \cdots & \cdots \\ \hline E & \\ \cline{1-1} \end{array}}.$$

The rule of variable elimination is:

$$\frac{\begin{array}{|r|c|}\hline \Gamma_0 & \Delta_0 \\ \hline x = t_1, \Gamma_1 & \Delta_1 \\ \cdots & \cdots \\ x = t_n, \Gamma_n & \Delta_n \\ \hline x = t_{n+1}, E & \\ \cline{1-1} \end{array}}{\begin{array}{|c|c|}\hline \Gamma_0 & \Delta_0 \\ \hline \Gamma_1 & \Delta_1 \\ \cdots & \cdots \\ \Gamma_n & \Delta_n \\ \hline E & \\ \cline{1-1} \end{array}}$$

where the variable x does not appear freely in the other part.

Remark 6.7 The rules of modalities realise the modality of S5. The rule of equation makes the axiom of distributivity sound.

Definition 6.8 (Theorem of the Gentzen-style system) For a formula P,

we write $\vdash_{\mathrm{G}} P$ when a sequent $\begin{array}{|c|c|}\hline & \\ \hline & P \\ \hline & \\ \cline{1-1} \end{array}$ is derived.

Theorem 6.9 $\vdash_{\mathrm{S5P}} P$ *iff* $\vdash_{\mathrm{G}} P$

7 Possible-world model

Definition 7.1 (Model)

1. A *model* is $M = (W, X)$, which is a pair of a set of *worlds* W and a set of *concepts* X.
2. A world is $w = (D_w, I_w) \in W$, that is, a world w consists of a set of *individuals* D_w and an *interpretation* I_w for the language Σ.
3. The set D_w, called an *individual domain*, is not empty.
4. For each predicate $p \in \Sigma$ of arity n, the interpretation I_w maps p into a subset $I_w(p) \subset D_w^n$. If '=' $\in \Sigma$, then the equality '=' is always mapped into the diagonal set, that is, $I_w(\text{'='}) = \{\langle d, d\rangle \mid d \in D_w\}$.
5. For each function symbol $f \in \Sigma$ of arity n, the interpretation I_w maps f into a function $I_w(f) : D_w^n \to D_w$. Each function symbol $f \in \Sigma$ of arity n has an action $\bar{f}$ over $\prod_{w \in W} D_w$ such as:

$$\text{For } \xi_1, \ldots, \xi_n \in \prod_{w \in W} D_w, \quad \bar{f}(\xi_1, \ldots, \xi_n)(w) = I_w(f)(\xi_1(w), \ldots, \xi_n(w)).$$

6. $X \subset \prod_{w \in W} D_w$ and X is closed under $\bar{f}$ for each function symbol $f \in \Sigma$, that is, if $\xi_1, \ldots, \xi_n \in X$ and f is a function symbol of arity n, then $\bar{f}(\xi_1, \ldots, \xi_n) \in X$.

Definition 7.2 (Environment) An *environment* is a map of variables V into concepts X. For an environment ρ, a variable $x \in V$ and a concept $\xi \in X$, we write $\rho[\xi/x]$ for another environment such as:
- $\rho[\xi/x](y) = \xi$ if y is x.
- $\rho[\xi/x](y) = \rho(y)$ if y is a variable other than x.

Remark 7.3 An environment of this definition maps a variable not into an individual but into a function of worlds into individuals. Hughes and Clesswell discuss such kind of environment for variables (Sec. 4, Chap 11 in [HC]). However, they do not give the axiomatisation nor the precise characterisation.

Definition 7.4 (Interpretation of terms) For an interpretation I_w and an environment ρ, the interpretation of terms is defined by induction on the terms as follows:
1. $I_{w\rho}(x) = \rho(x)(w)$ for $x \in V$
2. $I_{w\rho}(f(t_1, t_2, \ldots, t_n)) = I_w(f)(I_{w\rho}(t_1), I_{w\rho}(t_2), \ldots, I_{w\rho}(t_n))$

Definition 7.5 (Interpretation of formulae) For a model $M = (W, X)$, a world $w = (D_w, I_w) \in W$ and an environment $\rho : V \to X$, the interpretation of formulae is defined by induction on the formulae as follows:
1. For an atomic formula $p(t_1, t_2, \ldots, t_n)$,
   $(M, w, \rho) \models p(t_1, t_2, \ldots, t_n)$ iff $\langle I_{w\rho}(t_1), \ldots, I_{w\rho}(t_n)\rangle \in I_w(p)$
2. $(M, w, \rho) \models P \wedge Q$ iff $(M, w, \rho) \models P$ and $(M, w, \rho) \models Q$
3. $(M, w, \rho) \models \neg P$ iff $(M, w, \rho) \not\models P$
4. $(M, w, \rho) \models \forall x.P$ iff for all $\xi \in X$, $(M, w, \rho[\xi/x]) \models P$
5. $(M, w, \rho) \models \Box P$ iff for all $v \in W$, $(M, v, \rho) \models P$

We write $M \models P$ if $(M, w, \rho) \models P$ for all $w \in W$ and $\rho : V \to X$.

Definition 7.6 (Total abstraction model) A model $M = (W, X)$ is a *total abstraction model* if the followings hold:

1. For each $w \in W$, there are infinitely many worlds $v \in W$ which are isomorphic to w.
2. $X = \prod_{w \in W} D_w$

A formula P is valid for total abstraction iff $M \models P$ for all the total abstraction models M, and we write $\models_{\mathrm{T}} P$.

Definition 7.7 (Partial abstraction model) A model $M = (W, X)$ is a *partial abstraction model* if the followings hold:

1. For each $w \in W$, there are infinitely many worlds $v \in W$ which are isomorphic to w.
2. $M \models (\Box\exists x.P) \supset \exists x.\Box P$ for each classical formula P. In other words, for each classical formula P and each environment ρ, if it holds $(D_w, I_w, \rho) \models \exists x.P$ for each world $w \in W$, then there is a concept $\xi \in X$ such that $(D_w, I_w, \rho[\xi/x]) \models P$ for each world $w \in W$.
3. For each concept ξ, each world $w \in W$ and each individual $e \in D_w$, there is a concept $\xi' \in X$ such that
   - $\xi'(w) = e$
   - $\xi'(v) = \xi(v)$ for $v \neq w$

A formula P is valid for partial abstraction iff $M \models P$ for all the partial abstraction models M. and we write $\models_{\mathrm{P}} P$.

Remark 7.8 Let $M = (W, X)$ be a total abstraction model. If a world $w \in W$ of M has at least two individuals in D_w, then the set of concepts X cannot be countably many. That is because there are at least countably many worlds v's which are isomorphic to w. Thus each of D_v's has at least two individuals. Therefore X must be an uncountable set.

On the other hand, a partial abstraction model can be a countable model even if some worlds of it have plural elements.

Conjecture 7.9 *For each formula P, $\models_{\mathrm{T}} P$ iff $\models_{\mathrm{P}} P$.*

Theorem 7.10 (Soundness of S5D for the models) *If $\vdash_{\mathrm{G}} P$ then $\models_{\mathrm{T}} P$, thus $\models_{\mathrm{P}} P$.*

Theorem 7.11 (Completeness of S5D for partial abstraction models) *If $\models_{\mathrm{P}} P$ then P has a cut-free proof of $\vdash_{\mathrm{G}}$.*

Proof. By standard tableau method. ∎

Corollary 7.12 $\models_{\mathrm{L}} P$ *iff* $\models_{\mathrm{P}} P$

Corollary 7.13 *$\vdash_{\mathrm{G}}$ satisfies cut elimination.*

8 Conclusion

We have defined four systems of modal logic $\models_{\mathrm{L}}$, $\vdash_{\mathrm{S5P}}$, $\vdash_{\mathrm{G}}$ and $\models_{\mathrm{P}}$, and shown that all the systems are equivalent to each other. As the consequence, the Gentzen-style system $\vdash_{\mathrm{G}}$ satisfies cut elimination. The system $\models_{\mathrm{L}}$ is a semantical system defined in a purely syntactical way. On the other hand, the system $\models_{\mathrm{P}}$ is a semantical system defined by a variant of traditional possible-world semantics. These two systems present a striking contrast to each other, although they are equivalent to each other.

References

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Appendix

First of all we define the inequality which appears in structural rule.

– For sequences $\vec{x} = (x_1, x_2, ..., x_n)$ and $\vec{y} = (y_1, y_2, .., y_m)$, the inequality $\vec{x} \leq \vec{y}$ is defined as:

For each $x_i \in \{x_1, ..., x_n\}$, there is some $y_j \in \{y_1, ..., y_m\}$.

– For sequents $S$and S' such that

$$S = \begin{array}{|c|c|} \hline \Gamma_0 & \Delta_0 \\ \hline \Gamma_1 & \Delta_1 \\ \hline \dots & \dots \\ \hline \Gamma_n & \Delta_n \\ \hline E & \\ \hline \end{array}, \quad S' = \begin{array}{|c|c|} \hline \Gamma'_0 & \Delta'_0 \\ \hline \Gamma'_1 & \Delta'_1 \\ \hline \dots & \dots \\ \hline \Gamma'_m & \Delta'_m \\ \hline E' & \\ \hline \end{array},$$

the inequality $S \leq S'$ is defined as:

1. $\Gamma_0 \leq \Gamma'_0$, $\Delta_0 \leq \Delta'_0$
2. There is sequences $\Gamma''_1, \Gamma''_2, ..., \Gamma''_m, \Delta''_1, \Delta''_2, ..., \Delta''_m$, such that
   2.1 $(\langle \Gamma_1, \Delta_1 \rangle, \langle \Gamma_2, \Delta_2 \rangle, ..., \langle \Gamma_1, \Delta_1 \rangle) \leq (\langle \Gamma''_1, \Delta''_1 \rangle, \langle \Gamma''_2, \Delta''_2 \rangle, ..., \langle \Gamma''_1, \Delta''_1 \rangle)$
   2.2 $\Gamma''_i \leq \Gamma'_i$ and $\Delta''_i \leq \Delta'_i$ for each i.
3. $E \leq E'$

Initial rule: **Structural rule:**

$$\frac{}{\begin{array}{|c|c|}\hline & \\ \hline P & P \\ \hline \multicolumn{1}{|c|}{} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}} \qquad \frac{S}{S'} \text{ if } S < S'$$

¬-Left: **¬-Right:**

$$\frac{\begin{array}{|r|l|}\hline \Gamma_0 & \Delta_0 \\ \hline \cdots & \cdots \\ \Gamma_i & \Delta_i, P \\ \cdots & \cdots \\ \hline \multicolumn{1}{|c|}{E} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}{\begin{array}{|r|l|}\hline \Gamma_0 & \Delta_0 \\ \hline \cdots & \cdots \\ \neg P, \Gamma_i & \Delta_i \\ \cdots & \cdots \\ \hline \multicolumn{1}{|c|}{E} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}, \quad \frac{\begin{array}{|r|l|}\hline \Gamma_0 & \Delta_0, P \\ \hline \cdots & \cdots \\ \Gamma_i & \Delta_i \\ \cdots & \cdots \\ \hline \multicolumn{1}{|c|}{E} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}{\begin{array}{|r|l|}\hline \neg P, \Gamma_0 & \Delta_0 \\ \hline \cdots & \cdots \\ \Gamma_i & \Delta_i \\ \cdots & \cdots \\ \hline \multicolumn{1}{|c|}{E} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}} \qquad \frac{\begin{array}{|r|l|}\hline \Gamma_0 & \Delta_0 \\ \hline \cdots & \cdots \\ P, \Gamma_i & \Delta_i \\ \cdots & \cdots \\ \hline \multicolumn{1}{|c|}{E} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}{\begin{array}{|r|l|}\hline \Gamma_0 & \Delta_0 \\ \hline \cdots & \cdots \\ \Gamma_i & \Delta_i, \neg P \\ \cdots & \cdots \\ \hline \multicolumn{1}{|c|}{E} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}, \quad \frac{\begin{array}{|r|l|}\hline P, \Gamma_0 & \Delta_0 \\ \hline \cdots & \cdots \\ \Gamma_i & \Delta_i \\ \cdots & \cdots \\ \hline \multicolumn{1}{|c|}{E} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}{\begin{array}{|r|l|}\hline \Gamma_0 & \Delta_0, \neg P \\ \hline \cdots & \cdots \\ \Gamma_i & \Delta_i \\ \cdots & \cdots \\ \hline \multicolumn{1}{|c|}{E} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}$$

∧-Left:

$$\frac{\begin{array}{|r|l|}\hline \Gamma_0 & \Delta_0 \\ \hline \cdots & \cdots \\ P, Q, \Gamma_i & \Delta_i \\ \cdots & \cdots \\ \hline \multicolumn{1}{|c|}{E} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}{\begin{array}{|r|l|}\hline \Gamma_0 & \Delta_0 \\ \hline \cdots & \cdots \\ P \wedge Q, \Gamma_i & \Delta_i \\ \cdots & \cdots \\ \hline \multicolumn{1}{|c|}{E} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}, \quad \frac{\begin{array}{|r|l|}\hline P, Q, \Gamma_0 & \Delta_0 \\ \hline \cdots & \cdots \\ \Gamma_i & \Delta_i \\ \cdots & \cdots \\ \hline \multicolumn{1}{|c|}{E} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}{\begin{array}{|r|l|}\hline P \wedge Q, \Gamma_0 & \Delta_0 \\ \hline \cdots & \cdots \\ \Gamma_i & \Delta_i \\ \cdots & \cdots \\ \hline \multicolumn{1}{|c|}{E} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}$$

∧-Right:

$$\frac{\begin{array}{|r|l|}\hline \Gamma_0 & \Delta_0 \\ \hline \cdots & \cdots \\ \Gamma_i & \Delta_i, P \\ \cdots & \cdots \\ \hline \multicolumn{1}{|c|}{E} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|r|l|}\hline \Gamma_0 & \Delta_0 \\ \hline \cdots & \cdots \\ \Gamma_i & \Delta_i, Q \\ \cdots & \cdots \\ \hline \multicolumn{1}{|c|}{E} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}{\begin{array}{|r|l|}\hline \Gamma_0 & \Delta_0 \\ \hline \cdots & \cdots \\ \Gamma_i & \Delta_i, P \wedge Q \\ \cdots & \cdots \\ \hline \multicolumn{1}{|c|}{E} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}, \quad \frac{\begin{array}{|r|l|}\hline \Gamma_0 & \Delta_0, P \\ \hline \cdots & \cdots \\ \Gamma_i & \Delta_i \\ \cdots & \cdots \\ \hline \multicolumn{1}{|c|}{E} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|r|l|}\hline \Gamma_0 & \Delta_0, Q \\ \hline \cdots & \cdots \\ \Gamma_i & \Delta_i \\ \cdots & \cdots \\ \hline \multicolumn{1}{|c|}{E} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}{\begin{array}{|r|l|}\hline \Gamma_0 & \Delta_0, P \wedge Q \\ \hline \cdots & \cdots \\ \Gamma_i & \Delta_i \\ \cdots & \cdots \\ \hline \multicolumn{1}{|c|}{E} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}$$

$\forall$-Left:

$$
\frac{\begin{array}{|r|l|}\hline \Gamma_0 & \Delta_0 \\ \hline \cdots & \cdots \\ P[t/x], \Gamma_i & \Delta_i \\ \cdots & \cdots \\ \hline \multicolumn{1}{|r|}{E} & \\ \cline{1-1} \end{array}}{\begin{array}{|r|l|}\hline \Gamma_0 & \Delta_0 \\ \hline \cdots & \cdots \\ \forall x.P, \Gamma_i & \Delta_i \\ \cdots & \cdots \\ \hline \multicolumn{1}{|r|}{E} & \\ \cline{1-1} \end{array}}, \quad
\frac{\begin{array}{|r|l|}\hline P[t/x], \Gamma_0 & \Delta_0 \\ \hline \cdots & \cdots \\ \Gamma_i & \Delta_i \\ \cdots & \cdots \\ \hline \multicolumn{1}{|r|}{E} & \\ \cline{1-1} \end{array}}{\begin{array}{|r|l|}\hline \forall x.P, \Gamma_0 & \Delta_0 \\ \hline \cdots & \cdots \\ \Gamma_i & \Delta_i \\ \cdots & \cdots \\ \hline \multicolumn{1}{|r|}{E} & \\ \cline{1-1} \end{array}}
$$

$\forall$-Right:

$$
\frac{\begin{array}{|r|l|}\hline \Gamma_0 & \Delta_0 \\ \hline \cdots & \cdots \\ \Gamma_i & \Delta_i, P \\ \cdots & \cdots \\ \hline \multicolumn{1}{|r|}{E} & \\ \cline{1-1} \end{array}}{\begin{array}{|r|l|}\hline \Gamma_0 & \Delta_0 \\ \hline \cdots & \cdots \\ \Gamma_i & \Delta_i, \forall x.P \\ \cdots & \cdots \\ \hline \multicolumn{1}{|r|}{E} & \\ \cline{1-1} \end{array}}, \quad
\frac{\begin{array}{|r|l|}\hline \Gamma_0 & \Delta_0, P \\ \hline \cdots & \cdots \\ \Gamma_i & \Delta_i \\ \cdots & \cdots \\ \hline \multicolumn{1}{|r|}{E} & \\ \cline{1-1} \end{array}}{\begin{array}{|r|l|}\hline \Gamma_0 & \Delta_0, \forall x.P \\ \hline \cdots & \cdots \\ \Gamma_i & \Delta_i \\ \cdots & \cdots \\ \hline \multicolumn{1}{|r|}{E} & \\ \cline{1-1} \end{array}}
$$

x does not appear freely in the other parts.

$\Box$-Left:

$$
\frac{\begin{array}{|r|l|}\hline \Gamma_0 & \Delta_0 \\ \hline \cdots & \cdots \\ P, \Gamma_i & \Delta_i \\ \cdots & \cdots \\ \hline \multicolumn{1}{|r|}{E} & \\ \cline{1-1} \end{array}}{\begin{array}{|r|l|}\hline P, \Gamma_0 & \Delta_0 \\ \hline \cdots & \cdots \\ \Gamma_i & \Delta_i \\ \cdots & \cdots \\ \hline \multicolumn{1}{|r|}{E} & \\ \cline{1-1} \end{array}}, \quad
\frac{\begin{array}{|r|l|}\hline \Gamma_0 & \Delta_0 \\ \hline \cdots & \cdots \\ \Gamma_i & \Delta_i \\ \cdots & \cdots \\ \hline \multicolumn{1}{|r|}{t = t', E} & \\ \cline{1-1} \end{array}}{\begin{array}{|r|l|}\hline t = t', \Gamma_0 & \Delta_0 \\ \hline \cdots & \cdots \\ \Gamma_i & \Delta_i \\ \cdots & \cdots \\ \hline \multicolumn{1}{|r|}{E} & \\ \cline{1-1} \end{array}}, \quad
\frac{\begin{array}{|r|l|}\hline P, \Gamma_0 & \Delta_0 \\ \hline \cdots & \cdots \\ \Gamma_i & \Delta_i \\ \cdots & \cdots \\ \hline \multicolumn{1}{|r|}{E} & \\ \cline{1-1} \end{array}}{\begin{array}{|r|l|}\hline \Gamma_0 & \Delta_0 \\ \hline \cdots & \cdots \\ \Box P, \Gamma_i & \Delta_i \\ \cdots & \cdots \\ \hline \multicolumn{1}{|r|}{E} & \\ \cline{1-1} \end{array}}
$$

□-Right:

$$\dfrac{\begin{array}{|c|c|}\hline \Gamma_0 & \Delta_0 \\ \hline \Gamma_1 & \Delta_1 \\ \cdots & \cdots \\ \Gamma_n & \Delta_n \\ E & P \\ \hline E & \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline \Gamma_0 & \Delta_0, P \\ \hline \cdots & \cdots \\ \Gamma_i & \Delta_i \\ \cdots & \cdots \\ \hline E & \\ \hline \end{array}}{\begin{array}{|c|c|}\hline \Gamma_0 & \Delta_0, \Box P \\ \hline \Gamma_1 & \Delta_1 \\ \cdots & \cdots \\ \Gamma_n & \Delta_n \\ \hline E & \\ \hline \end{array} \quad , \quad \begin{array}{|c|c|}\hline \Gamma_0 & \Delta_0 \\ \hline \cdots & \cdots \\ \Gamma_i & \Delta_i, \Box P \\ \cdots & \cdots \\ \hline E & \\ \hline \end{array}}$$

Substitution:

$$\dfrac{\begin{array}{|r|l|}\hline \Gamma_0[t/x] & \Delta_0[t/x] \\ \hline \Gamma_1[t/x] & \Delta_1[t/x] \\ \cdots & \cdots \\ \Gamma_n[t/x] & \Delta_n[t/x] \\ \hline E[t/x] & \\ \hline \end{array} \qquad \begin{array}{|r|l|}\hline \Gamma_0 & \Delta_0 \\ \hline \Gamma_1 & \Delta_1 \\ \cdots & \cdots \\ \Gamma_n & \Delta_n \\ \hline E[t/x] & \\ \hline \end{array}}{\begin{array}{|r|l|}\hline t = t', \Gamma_0[t'/x] & \Delta_0[t'/x] \\ \hline \Gamma_1[t'/x] & \Delta_1[t'/x] \\ \cdots & \cdots \\ \Gamma_n[t'/x] & \Delta_n[t'/x] \\ \hline E[t'/x] & \\ \hline \end{array} \quad , \quad \begin{array}{|r|l|}\hline \Gamma_0 & \Delta_0 \\ \hline \Gamma_1 & \Delta_1 \\ \cdots & \cdots \\ \Gamma_n & \Delta_n \\ \hline t = t', E[t'/x] & \\ \hline \end{array}}$$

$$\dfrac{\begin{array}{|r|l|}\hline \Gamma_0 & \Delta_0 \\ \hline \cdots & \cdots \\ \Gamma_i[t/y] & \Delta_i[t/y] \\ \cdots & \cdots \\ \hline E & \\ \hline \end{array}}{\begin{array}{|r|l|}\hline \Gamma_0 & \Delta_0 \\ \hline \cdots & \cdots \\ t = t', \Gamma_i[t/y] & \Delta_i[t'/y] \\ \cdots & \cdots \\ \hline E & \\ \hline \end{array}}$$

y does not appear freely in any scopes of $\Box$.

Variable elimination:

$$\frac{\begin{array}{|r|l|}\hline x=t,\Gamma_0 & \Delta_0 \\ \hline \Gamma_1 & \Delta_1 \\ \dots & \dots \\ \Gamma_n & \Delta_n \\ \hline E & \\ \hline \end{array}}{\begin{array}{|c|c|}\hline \Gamma_0 & \Delta_0 \\ \hline \Gamma_1 & \Delta_1 \\ \dots & \dots \\ \Gamma_n & \Delta_n \\ \hline E & \\ \hline \end{array}}, \quad \frac{\begin{array}{|r|l|}\hline \Gamma_0 & \Delta_0 \\ \hline x=t_1,\Gamma_1 & \Delta_1 \\ \dots & \dots \\ x=t_n,\Gamma_n & \Delta_n \\ \hline x=t_{n+1},E & \\ \hline \end{array}}{\begin{array}{|c|c|}\hline \Gamma_0 & \Delta_0 \\ \hline \Gamma_1 & \Delta_1 \\ \dots & \dots \\ \Gamma_n & \Delta_n \\ \hline E & \\ \hline \end{array}}$$

x does not appear freely in the other parts.

Cut:

$$\frac{\begin{array}{|c|l|}\hline \Gamma_0 & \Delta_0 \\ \hline \dots & \dots \\ \Gamma_i & \Delta_i, P \\ \dots & \dots \\ \hline E & \\ \hline \end{array} \quad \begin{array}{|r|c|}\hline \Gamma_0 & \Delta_0 \\ \hline \dots & \dots \\ P,\Gamma_i & \Delta_i \\ \dots & \dots \\ \hline E & \\ \hline \end{array}}{\begin{array}{|c|c|}\hline \Gamma_0 & \Delta_0 \\ \hline \dots & \dots \\ \Gamma_i & \Delta_i \\ \dots & \dots \\ \hline E & \\ \hline \end{array}}$$